

The dynamics of the gravitational two-body problem in the post-Newtonian approximation at quadratic order in the Newton's constant

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We derive the conservative part of the Lagrangian and the energy of a gravitationally bound two-body system at fourth post-Newtonian order, up to terms quadratic in the Newton constant. We also show that such terms are compatible with Lorentz invariance and we write an ansatz for the center-of mass position. The remaining terms carrying higher powers of the Newton constant are currently under investigation.

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I. INTRODUCTION

The post-Newtonian approximation to the 2-body problem in General relativity represents the common approach to study the bound object dynamics in the weak curvature, slow motion regime, see [1] and [2] for reviews. The interest in detailed analytical study of the gravitationally bound two-body problem has been revived by several concurrent factors.

On the experimental side the era of gravitational wave astronomy is expected to start in a few years by the advent of the advanced gravitational wave detectors LIGO and Virgo [3]. Among other signals, these detectors will be sensitive to gravitational waves emitted

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by coalescing binaries, making urgent to derive the equation of motion of binary black holes with the highest possible accuracy. The knowledge of accurate template waveform is needed for matching filtering techniques, commonly used in data analysis, whose output is particularly sensitive to the time varying phase of the signal which must be computed with $O(1)$ precision [4].

On the theoretical side, since few years now stable numerical-relativity (NR) waveforms emitted in the last $O(10)$ orbits of a binary system (plus merger and ringdown to a final Kerr black hole) are available, see e.g. [5] for a recent review. As NR waveforms cannot be extended too early in the *inspiral* phase, during which binary constituents are far apart and non-relativistic, it appears natural to combine PN and NR results to construct complete waveform models: the highest possible accuracy on the analytical PN side is required in order to reduce the number of cycles of the NR waveforms needed to build complete, reliable waveforms [6].

On the theoretical analytical side, before the present paper, the Hamiltonian ruling the conservative dynamics of a gravitationally bound, spin-less binary system, has been computed at third PN order in [7] in the Arnowitt-Deser-Misner coordinates, calculation later confirmed by [8–10] in harmonic coordinates, and by [11], where a different resolution of the source singularity has been adopted. More recently, using the effective field theory (EFT) approach developed in [12], see [13] for a review, the Lagrangean of the gravitationally bound two body system has been re-derived at third PN order in [14], with the use of an algorithm automatizing the decomposition of the problem into the sum of terms associated to Feynman diagrams. The application of the EFT methods to the PN approximation of General Relativity has also made possible the derivation of new results for spinning systems in both the conservative [15] and dissipative dynamics [16].

Following on our previous work [14], the main result of this paper is the computation of the two-body effective action for the conservative dynamics at 4PN order and up to terms quadratic in Newton's constant G_N , and thus represents a decisive step towards the computation of the full Hamiltonian dynamics at 4PN order. Using the virial relation $v^2 \sim G_N M/r$, being $r(v)$ the relative distance (velocity) of the binary constituents with M the total mass, the terms contributing to the 4PN order can be parametrized as $G_N^{5-n} v^{2n}$ with $0 \leq n \leq 4$.

The remaining terms at 4PN not presented here involve 569 Feynman graphs whose

computation requires the evaluation of integrals among which the most complicated ones are few integrals analogous to 4-loop momentum integral in quantum field theory. The computation of the full Lagrangean at 4PN order is well under way and will be the subject of a future publication.

We also note that the 4PN conservative contribution of the back-reaction force has been computed in [17] and re-derived within EFT methods in [18].

The paper is organized as follows. In sec. II the fundamental gravity Lagrangean relevant for the two-body dynamics in the PN approximation is laid down and in sec. III we give a brief overview of the effective field theory methods for gravity, which are applied in the rest of the paper to the evaluation of the relevant Feynman graphs giving contribution to the 4PN order up G_N^2 . In sec. IV standard techniques are applied to get rid of terms quadratic in the accelerations, in order to obtain a Lagrangean with terms at most linear in the acceleration. In sec. V the energy is derived and in sec. VI the Lorentz invariance is checked for and the center-of-mass position of the two-body system derived, always at $O(G^2)$ order. We summarize and conclude in sec. VII.

II. SHORT-SCALE LAGRANGIAN

We perform our calculation in the EFT framework along the lines of [14]; in this section we resume our notations. The starting point is the action

$$S = S_{EH} + S_{GF} + S_{pp}, \quad (1)$$

the first and third terms being, respectively, the usual Einstein-Hilbert action[25] and the worldline point particle action

$$S_{EH} = 2\Lambda^2 \int d^{d+1}x \sqrt{-g} R(g), \quad S_{pp} = -\sum_{i=1,2} m_i \int d\tau_i = -\sum_{i=1,2} m_i \int \sqrt{-g_{\mu\nu}(x_i^\mu) dx_i^\mu dx_i^\nu}, \quad (2)$$

with

$$\Lambda^{-2} \equiv 32\pi G_N L^{d-3}. \quad (3)$$

As dimensional regularization will be needed in the computation, L is a reference length scale which must cancel out from physical results and G_N is the standard 3+1-dimensional

Newton constant. As to the gauge fixing term S_{GF} , we follow [19]:

$$S_{GF} = -\Lambda^2 \int d^{d+1}x \sqrt{-g} \Gamma_\mu \Gamma^\mu, \quad (4)$$

with $\Gamma^\mu \equiv \Gamma_{\alpha\beta}^\mu g^{\alpha\beta}$, which corresponds to the same harmonic gauge adopted in [1]. Still following [19], we adopt the standard Kaluza-Klein (KK) parametrization of the metric [20] (a somehow similar parametrization was first applied within the framework of a PN calculation in [21]):

$$g_{\mu\nu} = e^{2\phi/\Lambda} \begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d\phi/\Lambda} \gamma_{ij} - A_i A_j / \Lambda^2 \end{pmatrix}, \quad (5)$$

with $\gamma_{ij} = \delta_{ij} + \sigma_{ij}/\Lambda$, $c_d = 2\frac{(d-1)}{(d-2)}$ and i, j running over the d spatial dimensions. In terms of the metric parametrization (5), each world-line coupling to the gravitational degrees of freedom ϕ , A_i , σ_{ij} reads

$$S_{pp} = -m \int d\tau = -m \int dt e^{\phi/\Lambda} \sqrt{\left(1 - \frac{A_i}{\Lambda} v^i\right)^2 - e^{-c_d\phi/\Lambda} \left(v^2 + \frac{\sigma_{ij}}{\Lambda} v^i v^j\right)}, \quad (6)$$

and its Taylor expansion provides the various particle-gravity vertices of the EFT.

Also the pure gravity sector $S_{bulk} = S_{EH} + S_{GF}$ can be explicitly written in terms of the KK variables; we report here only those terms which are needed for the full 4PN calculation (only a part of which is performed in the present work):

$$\begin{aligned} \frac{S_{bulk}}{\Lambda^2} \supset & \int d^{d+1}x \sqrt{-\gamma} \left\{ \frac{1}{4} \left[(\vec{\nabla}\sigma)^2 - 2(\vec{\nabla}\sigma_{ij})^2 - (\dot{\sigma}^2 - 2(\dot{\sigma}_{ij})^2) e^{-\frac{c_d\phi}{\Lambda}} \right] - c_d \left[(\vec{\nabla}\phi)^2 - \dot{\phi}^2 e^{-\frac{c_d\phi}{\Lambda}} \right] \right. \\ & + \left[\frac{F_{ij}^2}{2} + (\vec{\nabla}\cdot\vec{A})^2 - \dot{\vec{A}}^2 e^{-\frac{c_d\phi}{\Lambda}} \right] e^{\frac{c_d\phi}{\Lambda}} + \frac{2}{\Lambda} \left[(F_{ij} A^i \dot{A}^j + \vec{A}\cdot\dot{\vec{A}}(\vec{\nabla}\cdot\vec{A})) e^{\frac{c_d\phi}{\Lambda}} - c_d \dot{\phi} \vec{A}\cdot\vec{\nabla}\phi \right] \\ & + 2c_d \left(\dot{\phi} \vec{\nabla}\cdot\vec{A} - \dot{\vec{A}}\cdot\vec{\nabla}\phi \right) + \frac{\dot{\sigma}_{ij}}{\Lambda} \left(-\delta^{ij} A_l \hat{\Gamma}_{kk}^l + 2A_k \hat{\Gamma}_{ij}^k - 2A^i \hat{\Gamma}_{kk}^j \right) - c_d \frac{\dot{\phi}^2 \vec{A}^2}{\Lambda^2} \\ & \left. - \frac{1}{\Lambda} \left(\frac{\sigma}{2} \delta^{ij} - \sigma^{ij} \right) (\sigma_{ik}{}^{,l} \sigma_{jl}{}^{,k} - \sigma_{ik}{}^{,k} \sigma_{jl}{}^{,l} + \sigma_{,i} \sigma_{jk}{}^{,k} - \sigma_{ik,j} \sigma^{,k}) \right\}. \end{aligned} \quad (7)$$

The form (7) of the gravitational action [26] is in agreement with the one derived in [20].

III. INTEGRATING OUT GRAVITY

We have now all the ingredients to obtain a 2-body effective action S_{eff} with manifest power counting in G_N and v at the desired (fourth) post-Newtonian order. This can be done

by integrating out the graviton fields from the full action derived above

$$S_{eff} = \int D\phi D\sigma_{ij} DA_k \exp[i(S_{EH} + S_{GF} + iS_{pp})], \quad (8)$$

to obtain an effective action S_{eff} in terms of the particle positions and thier derivatives. As usual in field theory, the functional integration can be perturbatively expanded in terms of Feynman diagrams involving the gravitational degrees of freedom as internal lines [27], regarded as dynamical fields emitted and absorbed by the point particles which are taken as non-dynamical sources.

In order to allow manifest v scaling it is convenient to work with the space-Fourier transformed fields

$$W_{\mathbf{p}}^a(t) \equiv \int d^d x W^a(t, x) e^{-i\mathbf{p}\cdot\mathbf{x}} \quad \text{with } W^a = \{\phi, A_i, \sigma_{ij}\}. \quad (9)$$

The fields defined above are the fundamental variables in terms of which we are going to construct the Feynmann graphs; the action governing their dynamics can be found from eqns. (6,7).

By looking in particular at the quadratic parts, one can explicitly write the propagators:

$$P[W_{\mathbf{p}}^a(t_a) W_{\mathbf{p}'}^b(t_b)] = \frac{1}{2} P^{aa} \delta_{ab} (2\pi)^d \delta^d(\mathbf{p} + \mathbf{p}') \mathcal{P}(\mathbf{p}^2, t_a, t_b) \delta(t_a - t_b), \quad (10)$$

where $P^{\phi\phi} = -\frac{1}{c_d}$, $P^{A_i A_j} = \delta_{ij}$, $P^{\sigma_{ij} \sigma_{kl}} = -(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} + (2 - c_d) \delta_{ij} \delta_{kl})$ and

$$\mathcal{P}(\mathbf{p}^2, t_a, t_b) = \frac{i}{\mathbf{p}^2 - \partial_{t_a} \partial_{t_b}} \simeq \frac{i}{\mathbf{p}^2} \left(1 + \frac{\partial_{t_a} \partial_{t_b}}{\mathbf{p}^2} + \frac{\partial_{t_a}^2 \partial_{t_b}^2}{\mathbf{p}^4} \dots \right) \quad (11)$$

is the full relativistic propagator, which have been expanded as an instantaneaus nonrelativistic part plus insertion terms involving time derivatives (which after hitting the $e^{i\mathbf{p}\cdot\mathbf{x}}$ factors bring in extra v factors) and the infinite series is truncated at the some finite order in every Feynmann diagram.

The next step is to to lay down and compute all the relevant Feynman diagrams; this has been done by means of a new version of the Mathematica code [22], including the use of the Feyncalc software [23], which has been used to reproduce the 3PN Lagrangian [14].



Figure 1: The three diagrams contributing at order G_N . The ϕ , A and σ propagators are represented respectively by blue dashed, red dotted and green solid lines. Propagators and vertices must be considered at appropriate v order to pick the 4PN contribution of these graphs.

A. Order G_N

The three relevant diagrams at this order are shown in fig. 1, the typical graph contribution A to the effective action (8) has the following form:

$$A \sim G_N m_1 m_2 L^{d-3} \int_{\mathbf{p}, t_1, t_2} \sum_n \frac{(\partial_{t_1} \partial_{t_2})^n}{\mathbf{p}^{2(n+1)}} e^{i\mathbf{p} \cdot (x_1 - x_2)} V_1 V_2, \quad (12)$$

where integration is performed over times $t_{1,2}$ and the measure of momentum integration is $d^d p / (2\pi)^d$, n is an integer number with $n \leq 4$, $x_{1,2} \equiv x_{1,2}(t_{1,2})$, and $V_{1,2} \equiv V_{1,2}[v_{1,2}(t_{1,2})]$ are the appropriate point particle vertices, expanded at the appropriate order in $v_{1,2}$, which can be read from eq. (6). The integrations can be performed via standard Fourier formulae and do not lead to any divergency in dimensional regularization.

We report below the result of the sum of the diagrams; symmetrization under exchange

of particles $+(1 \leftrightarrow 2)$ is understood throughout all this paper

$$\begin{aligned}
\mathcal{L}_{G_N}^{4PN} = & \frac{G_N m_1 m_2}{r} \left\{ v_2 \cdot a_1 \left[\frac{3}{16} (v_1 \cdot v_2)^2 v_1^r - v_2^4 \left(\frac{49}{64} v_1^r + \frac{123}{64} v_2^r \right) - \frac{33}{32} v_1^2 v_2^2 v_2^r \right. \right. \\
& + v_2^2 v_1 \cdot v_2 \left(\frac{1}{16} v_1^r + \frac{3}{16} v_2^r \right) + v_2^2 \left(\frac{17}{96} v_1^{r3} + \frac{7}{32} v_1^{r2} v_2^r + \frac{13}{32} v_1^r v_2^{r2} + \frac{77}{96} v_2^{r3} \right) \\
& + v_1 \cdot v_2 \left(\frac{1}{16} v_1^{r2} + \frac{5}{32} v_1^r v_2^r + \frac{5}{32} v_2^{r2} \right) v_1^r - v_1^{r3} \left(\frac{13}{64} v_1^{r2} + \frac{11}{64} v_1^r v_2^r + \frac{5}{32} v_2^{r2} \right) \Big] r \\
& + v_1 \cdot a_1 v_2^2 \left[v_1^2 \left(\frac{123}{32} v_1^r + \frac{93}{32} v_2^r \right) - \frac{33}{16} v_1 \cdot v_2 v_2^r - v_2^{r2} \left(\frac{23}{32} v_1^r + \frac{77}{96} v_2^r \right) \right] r \\
& + r a_1^r \left[\frac{123}{128} v_1^4 v_2^2 + \frac{1}{16} (v_1 \cdot v_2)^3 + \frac{1}{32} v_2^2 (v_1 \cdot v_2)^2 - \frac{49}{64} v_2^4 v_1 \cdot v_2 + \frac{75}{128} v_2^6 - \frac{23}{64} v_1^2 v_2^2 v_2^{r2} \right. \\
& + (v_1 \cdot v_2)^2 \left(\frac{3}{32} v_1^{r2} + \frac{5}{32} v_1^r v_2^r + \frac{5}{64} v_2^{r2} \right) + v_2^2 v_1 \cdot v_2 \left(\frac{17}{32} v_1^{r2} + \frac{7}{16} v_1^r v_2^r + \frac{13}{32} v_2^{r2} \right) \\
& - v_2^4 \left(\frac{57}{128} v_1^{r2} + \frac{31}{64} v_1^r v_2^r + \frac{53}{128} v_2^{r2} \right) - v_1 \cdot v_2 \left(\frac{65}{64} v_1^{r2} + \frac{11}{16} v_1^r v_2^r + \frac{15}{32} v_2^{r2} \right) v_1^{r2} \\
& + v_2^2 \left(\frac{65}{128} v_1^{r4} + \frac{15}{32} v_1^{r3} v_2^r + \frac{27}{64} v_1^{r2} v_2^{r2} + \frac{11}{32} v_1^r v_2^{r3} + \frac{27}{128} v_2^{r4} \right) \\
& \left. - v_2^{r4} \left(\frac{15}{128} v_1^{r2} + \frac{5}{64} v_1^r v_2^r + \frac{5}{128} v_2^{r2} \right) \right] \\
& + \frac{75}{128} v_1^8 - \frac{1}{32} (v_1 \cdot v_2)^4 + \frac{203}{128} v_1^6 v_2^2 - \frac{5}{4} v_1^6 v_1 \cdot v_2 + \frac{1}{16} (v_1 \cdot v_2)^3 v_1^2 + \frac{3}{256} v_1^4 v_2^4 \\
& + \frac{3}{4} v_1^4 (v_1 \cdot v_2)^2 - \frac{11}{32} v_1^6 v_2^{r2} - \frac{111}{64} v_1^4 v_2^2 v_1 \cdot v_2 - v_1^4 v_2^2 \left(\frac{57}{128} v_2^{r2} + \frac{123}{128} v_1^{r2} - \frac{15}{64} v_1^r v_2^r \right) \\
& + \frac{3}{4} v_1^4 v_1 \cdot v_2 v_2^{r2} - (v_1 \cdot v_2)^3 \left(\frac{5}{64} v_1^r + \frac{1}{32} v_2^r \right) v_1^r \\
& + (v_1 \cdot v_2)^2 v_1^2 \left(\frac{3}{32} v_1^{r2} + \frac{5}{32} v_1^r v_2^r - \frac{27}{64} v_2^{r2} \right) + v_1^2 v_1 \cdot v_2 v_2^2 \left(\frac{17}{32} v_1^r + \frac{3}{8} v_2^r \right) v_1^r \\
& + (v_1 \cdot v_2)^2 \left(\frac{107}{128} v_1^{r4} + \frac{25}{64} v_1^{r3} v_2^r + \frac{15}{64} v_1^{r2} v_2^{r2} - \frac{5}{32} v_1^r v_2^{r3} - \frac{11}{128} v_2^{r4} \right) \\
& + \frac{21}{64} v_1^4 v_2^{r4} - v_1^2 v_1 \cdot v_2 \left(\frac{65}{64} v_1^{r4} + \frac{11}{16} v_1^{r3} v_2^r + \frac{15}{32} v_1^{r2} v_2^{r2} + \frac{3}{4} v_2^{r4} \right) \\
& + v_1^2 v_2^2 \left(\frac{65}{128} v_1^{r2} + \frac{15}{32} v_1^r v_2^r + \frac{3}{4} v_2^{r2} \right) v_1^{r2} - v_1^2 v_2^{r4} \left(\frac{15}{128} v_1^{r2} + \frac{5}{64} v_1^r v_2^r + \frac{65}{128} v_2^{r2} \right) \\
& + v_1 \cdot v_2 \left(\frac{65}{64} v_1^{r3} - \frac{5}{32} v_1^{r2} v_2^r - \frac{5}{64} v_1^r v_2^{r2} - \frac{15}{32} v_2^{r3} \right) v_1^{r3} + \frac{35}{256} v_1^{r4} v_2^{r4} \Big\} + \mathcal{L}_{G_N}^{a^2}, \quad (13)
\end{aligned}$$

where $r \equiv |x_1 - x_2|$, $v_{1,2}^r \equiv v_{1,2} \cdot (x_1 - x_2)/r$, $a_{1,2}^r \equiv a_{1,2} \cdot (x_1 - x_2)/r$, and terms depending

quadratically on accelerations and their derivatives have been isolated in

$$\begin{aligned}
\mathcal{L}_{G_N}^{a^2} = & G_N m_1 m_2 r \left\{ \left[\frac{31}{11520} \ddot{a}_1 \cdot \ddot{a}_2 - \frac{1}{2304} \ddot{a}_1^r \ddot{a}_2^r \right] r^4 + \left[\frac{1}{96} \ddot{a}_1^r \ddot{a}_2^r v_1^r - \frac{7}{288} \ddot{a}_1^r v_{12} \cdot \dot{a}_2 - \frac{23}{288} \dot{a}_2 \cdot \ddot{a}_1 v_1^r \right] r^3 \right. \\
& + \left[\left(\frac{1}{32} v_1 \cdot v_2 - \frac{133}{576} v_1^2 - \frac{91}{576} a_1^r r - \frac{91}{192} v_1^{r2} - \frac{11}{96} v_1^r v_2^r \right) \dot{a}_1 \dot{a}_2 + \left(\frac{11}{192} v_1^2 - \frac{1}{32} v_1 \cdot v_2 + \frac{5}{192} a_1^r r \right. \right. \\
& + \left. \left. \frac{5}{192} v_1^{r2} + \frac{1}{96} v_1^r v_2^r \right) \dot{a}_1^r \dot{a}_2^r + \left(\frac{9}{32} v_1 \cdot \dot{a}_1 v_2^r + \frac{1}{6} v_1 \cdot \dot{a}_1 v_1^r - \frac{7}{48} v_2 \cdot \dot{a}_1 v_1^r + \frac{59}{576} a_2 \cdot \dot{a}_1 r + \frac{3}{64} a_1 \cdot \dot{a}_1 r \right) \dot{a}_2^r \right. \\
& + \left. \frac{1}{32} v_2 \cdot \dot{a}_1 v_1 \cdot \dot{a}_2 - \frac{2}{9} v_1 \cdot \dot{a}_1 v_1 \cdot \dot{a}_2 + \frac{1}{9} v_1 \cdot \dot{a}_1 v_2 \cdot \dot{a}_2 \right] r^2 + \left[\left(\frac{59}{64} a_1^r v_1^r r + \frac{29}{64} a_1^r v_2^r r - \frac{19}{192} v_2 \cdot a_1 r \right. \right. \\
& + \left. \frac{37}{64} v_1 \cdot a_1 r + \frac{85}{64} v_1^2 v_1^r + \frac{55}{64} v_1^2 v_2^r + \frac{5}{32} v_1 \cdot v_2 v_1^r + \frac{59}{384} (v_1^{r3} + v_2^{r3}) + \frac{29}{128} (v_1^r + v_2^r) v_1^r v_2^r \right) a_2 \cdot \dot{a}_1 \\
& + \left(\frac{37}{64} a_1 \cdot a_2 r + \frac{67}{384} a_2^2 r + \left\{ \frac{27}{64} v_1 \cdot a_2 - \frac{145}{64} v_2 \cdot a_1 - \frac{59}{64} v_2 \cdot a_2 \right\} v_1^r \right. \\
& + \left. \left\{ \frac{13}{64} v_1 \cdot a_2 - \frac{27}{64} v_2 \cdot a_1 - \frac{35}{64} v_2 \cdot a_2 \right\} v_2^r \right) v_1 \cdot \dot{a}_1 + \left(\left\{ \frac{37}{64} v_2 \cdot a_1 + \frac{53}{64} v_2 \cdot a_2 - \frac{145}{64} v_1 \cdot a_1 - \frac{3}{64} v_1 \cdot a_2 \right\} v_1^r \right. \\
& + \left. \left\{ \frac{3}{64} (v_2 \cdot a_1 - v_1 \cdot a_2) - \frac{27}{64} v_1 \cdot a_1 + \frac{67}{64} v_2 \cdot a_2 \right\} v_2^r - \frac{19}{192} a_1 \cdot a_2 r - \frac{67}{384} a_2^2 r \right) v_2 \cdot \dot{a}_1 \Big] r \\
& + \left[\left(\left\{ \frac{59}{64} a_1 \cdot a_2 + \frac{27}{128} a_2^2 - \frac{5}{64} a_1^r a_2^r \right\} v_1^r + \left\{ \frac{29}{64} a_1 \cdot a_2 + \frac{29}{128} a_2^2 - \frac{3}{64} a_1^r a_2^r \right\} v_2^r \right. \right. \\
& + \left. \frac{27}{64} v_1 \cdot a_2 a_1^r + \frac{17}{64} v_2 \cdot a_2 a_2^r - \frac{13}{64} v_1 \cdot a_2 a_2^r \right) r + \left(\frac{23}{64} v_2^2 + \frac{5}{32} v_1 \cdot v_2 + \frac{13}{64} v_2^{r2} + \frac{13}{32} v_1^r v_2^r \right) v_1 \cdot a_2 \\
& + \left(\frac{37}{64} v_1^2 + \frac{27}{64} v_1^{r2} \right) v_1 \cdot a_2 + \left(\frac{9}{32} v_1 \cdot v_2 - \frac{51}{64} v_2^2 - \frac{17}{64} v_2^{r2} - \frac{15}{32} v_1^r v_2^r \right) v_2 \cdot a_2 + \left(\frac{5}{32} v_1 \cdot v_2 v_1^r - \frac{11}{64} v_1^2 v_1^r \right. \\
& - \left. \frac{9}{64} v_1^2 v_2^r + \frac{5}{192} v_1^{r3} + \frac{3}{64} v_1^{r2} v_2^r \right) a_2^r \Big] r \dot{a}_1^r + \left[\left(\frac{37}{64} a_1^2 - \frac{19}{384} a_1 \cdot a_2 + \frac{145}{128} a_1^{r2} + \frac{1}{8} a_1^r a_2^r \right) a_1 \cdot a_2 \right. \\
& + \left. \frac{67}{768} a_1^2 a_2^2 - \frac{3}{256} a_1^{r2} a_2^{r2} - \frac{5}{128} a_1^{r3} a_2^r \right] r^2 + \left[\left(\frac{105}{64} v_1 \cdot a_1 + \frac{27}{64} v_1 \cdot a_2 - \frac{145}{64} v_2 \cdot a_1 - \frac{59}{64} v_2 \cdot a_2 \right) v_1^r \right. \\
& + \left(\frac{15}{64} v_1 \cdot a_1 + \frac{13}{64} v_1 \cdot a_2 - \frac{27}{64} v_2 \cdot a_1 - \frac{35}{64} v_2 \cdot a_2 \right) v_2^r \Big] r a_1^2 + \left[\frac{111}{32} v_1 \cdot a_1 v_1^r + \frac{7}{32} v_2^r v_1 \cdot a_1 - \frac{17}{32} v_2 \cdot a_1 v_1^r \right. \\
& - \left. \frac{5}{32} v_2 \cdot a_1 v_2^r + \left(\frac{181}{64} v_1^2 - \frac{5}{32} v_1 \cdot v_2 - \frac{3}{32} v_2^2 + \frac{231}{128} v_1^{r2} + \frac{25}{64} v_1^r v_2^r - \frac{3}{128} v_2^{r2} \right) a_1^r \right] r a_1 \cdot a_2 \\
& + \left[\left(\frac{105}{128} v_1 \cdot a_1 - \frac{5}{128} v_2 \cdot a_2 \right) v_1^r + \left(\frac{15}{128} v_1 \cdot a_1 - \frac{3}{128} v_2 \cdot a_2 \right) v_2^r + \left(\frac{65}{128} v_1^r + \frac{11}{128} v_2^r \right) (v_1 \cdot a_2 - v_2 \cdot a_1) \right. \\
& + \left. \left(\frac{11}{64} v_1 \cdot v_2 - \frac{27}{128} v_1^2 + \frac{3}{128} v_2^2 + \frac{15}{128} v_1^{r2} + \frac{5}{64} v_1^r v_2^r - \frac{3}{128} v_2^{r2} \right) a_2^r \right] r a_1^{r2} \\
& + \left[\frac{15}{64} v_2 \cdot a_1 v_1^r - \frac{27}{64} v_1 \cdot a_1 v_1^r + \frac{3}{64} v_2 \cdot a_1 v_2^r - \frac{21}{64} v_1 \cdot a_1 v_2^r \right] r a_1^r a_2^r + \left(\frac{3}{128} v_2 \cdot a_1 - \frac{27}{64} v_1 \cdot a_1 \right) v_2 \cdot a_1 r a_2^r \\
& + \left[\left(2 v_1 \cdot a_2 - \frac{145}{64} v_2 \cdot a_1 - \frac{5}{2} v_2 \cdot a_2 \right) v_1 \cdot a_1 + \left(\frac{37}{128} v_2 \cdot a_1 - \frac{5}{16} v_1 \cdot a_2 + \frac{49}{32} v_2 \cdot a_2 \right) v_2 \cdot a_1 \right] r a_1^r \\
& + \left[\frac{299}{128} v_1^4 - \frac{1}{4} (v_1 \cdot v_2)^2 - \frac{19}{32} v_1^2 v_1 \cdot v_2 + \frac{17}{64} v_1^2 v_2^2 + \left(\frac{303}{128} v_1^{r2} + \frac{45}{64} v_1^r v_2^r + \frac{17}{128} v_2^{r2} \right) v_1^2 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{5}{64}(v_1^r + v_2^r)v_1.v_2v_1^r - \left(\frac{43}{192}v_1^{r2} + \frac{5}{24}v_1^rv_2^r + \frac{13}{128}v_2^{r2}\right)v_1^{r2} \Big] a_1.a_2 + \left[\frac{67}{128}v_1^2v_1.v_2 - \frac{53}{128}v_1^4 - \frac{7}{64}v_1^2v_2^2\right. \\
& - \frac{15}{128}v_1.v_2v_2^2 + \left(\frac{27}{64}v_1^{r2} + \frac{11}{32}v_1^rv_2^r + \frac{3}{32}v_2^{r2}\right)v_1^2 - \left(\frac{11}{32}v_1^r + \frac{1}{8}v_2^r\right)v_1^rv_1.v_2 - \left(\frac{15}{128}v_1^r + \frac{5}{32}v_2^r\right)v_1^{r3} \Big] a_1^ra_2^r \\
& + \left[\left(\left\{\frac{19}{32}v_1.v_2 - \frac{23}{32}v_2^2 - \frac{73}{64}v_1^2\right\}v_1^r + \left\{\frac{17}{16}v_1.v_2 - \frac{123}{64}v_1^2\right\}v_2^r + \frac{9}{64}v_1^{r3} + \frac{21}{64}v_1^{r2}v_2^r + \frac{15}{32}v_1^rv_2^{r2}\right)v_1.a_1\right. \\
& + \left(\left\{\frac{25}{64}v_1^2 + \frac{5}{32}v_1.v_2 + \frac{13}{32}v_2^2\right\}v_1^r + \left\{\frac{43}{64}v_1^2 + \frac{3}{16}v_1.v_2\right\}v_2^r - \frac{5}{64}v_1^{r3} - \frac{13}{64}v_1^{r2}v_2^r - \frac{11}{32}v_1^rv_2^{r2}\right)v_2.a_1 \\
& + \left(\frac{65}{96}v_2^{r2} - \frac{77}{32}v_2^2\right)v_2^rv.a_1 \Big] a_2^r + \left[\frac{161}{32}v_1^2 - \frac{5}{8}v_1.v_2 + \frac{47}{16}v_2^2 + 2v_1^{r2} + \frac{13}{8}v_1^rv_2^r + \frac{49}{32}v_2^{r2}\right] v_1.a_1v_1.a_2 \\
& + \left[\frac{59}{32}(v_1.v_2 + v_1^rv_2^r) - \frac{145}{64}(v_1^2 + v_1^{r2}) + \frac{27}{64}(v_2^2 + v_2^{r2})\right] v_1.a_1v_2.a_1 + \left[\frac{119}{64}v_1.v_2 - \frac{177}{32}v_1^2 - \frac{5}{2}v_1^{r2}\right. \\
& - \frac{65}{64}v_1^rv_2^r \Big] v_1.a_1v_2.a_2 + \left[\frac{29}{32}v_1^2 - \frac{27}{64}v_1.v_2 - \frac{5}{16}v_1^{r2} - \frac{3}{64}v_1^rv_2^r\right] v_2.a_1v_1.a_2 \\
& + \left[\frac{37}{128}(v_1^2 + v_1^{r2}) - \frac{17}{64}(v_1.v_2 + v_1^rv_2^r) - \frac{3}{128}(v_2^2 + v_2^{r2})\right] (v_2.a_1)^2 \Big\} . \tag{14}
\end{aligned}$$

The rationale for this separation is that such terms can be further reduced via the *double zero trick* (see e.g. [24]) which consists in writing, for instance

$$\begin{aligned}
a_1 \cdot a_2 &= (a_1 - eq_1 + eq_1) \cdot (a_2 - eq_2 + eq_2) \\
&= (a_1 - eq_1) \cdot (a_2 - eq_2) + eq_1 \cdot a_2 + a_1 \cdot eq_2 - eq_1 \cdot eq_2 \\
&\equiv (a_1 - eq_1) \cdot (a_2 - eq_2) + (a_1 \cdot a_2)_{z^2} .
\end{aligned} \tag{15}$$

being $eq_{1,2}$ the value of $a_{1,2}$ in terms of r , v_1 and v_2 as dictated by the equations of motion at the desired PN order. Since the first term on the last line of eq. 15 gives a vanishing contribution to the equations of motion, it can be dropped from the Lagrangian, thus leaving only terms linear in the accelerations, which however carry higher powers of G_N , as $eq_{1,2} = \mathcal{O}(G_N)$. The standard form of the 3PN Lagrangian, as it is for instance reported in eq. (174) of [1], is obtained via reiterated use of this trick.

B. Order G_N^2

The 23 diagrams contributing at $\mathcal{O}(G_N^2)$ are shown in figure 2. These diagrams are either factorizable into two single-graviton exchanges, or give a contribution A to the effective

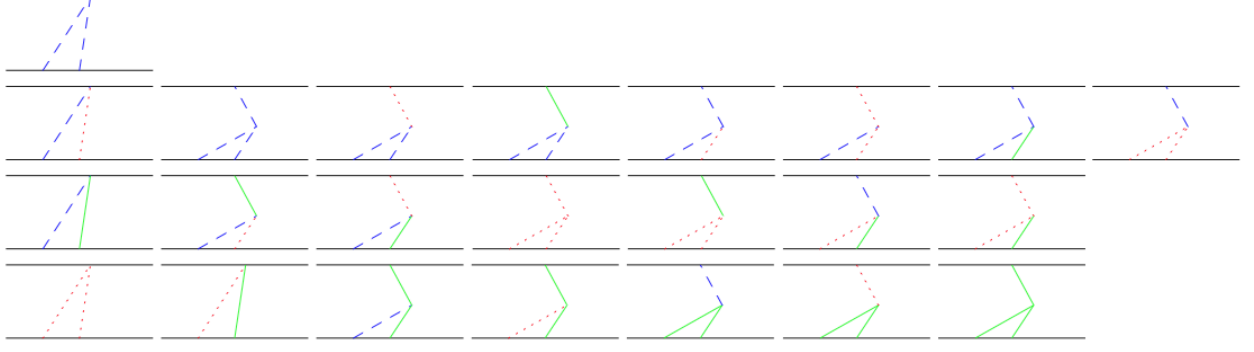


Figure 2: The diagrams contributing at order G_N^2 arranged in 4 lines. Diagrams in the n^{th} line enter the dynamics at $n\text{PN}$ order. Propagators and vertices must be considered at appropriate v order to pick the 4PN contribution of these diagrams.

action of the type:

$$\begin{aligned}
 A \sim & G_N^2 m_1^2 m_2 L^{2(d-3)} \sum_{n_a, n_b, n_2} \int_{t, t_{1a}, t_{1b}, t_2} (i\partial_{t_{1a}})^{2n_a} (i\partial_{t_{1b}})^{2n_b} (i\partial_{t_2})^{2n_2} V_{1a} V_{1b} V_2 \\
 & \times \int_{\mathbf{p}, \mathbf{q}} \frac{e^{i[(\mathbf{p}-\mathbf{q}) \cdot \mathbf{x}_1(t_{1a}) + \mathbf{q} \cdot \mathbf{x}_1(t_{1b}) - \mathbf{p} \cdot \mathbf{x}_2(t_2)]}}{(\mathbf{p}-\mathbf{q})^{2(n_a+1)} \mathbf{q}^{2(n_b+1)} \mathbf{p}^{2(n_2+1)}} V_{bulk} \delta(t-t_{1a}) \delta(t-t_{1b}) \delta(t-t_2), \quad (16)
 \end{aligned}$$

with the $n_i \leq 3$ and $V_{bulk} \equiv V_{bulk}[\mathbf{p}, \mathbf{q}, \partial_{t_{1a}}, \partial_{t_{1b}}, \partial_{t_2}]$ being the bulk vertices describing the appropriate three-graviton interaction.

These integrals can be reconduced to standard ones, eventually after some integration by parts, leading to the following total result:

$$\begin{aligned}
\mathcal{L}_{G_N^2}^{4PN} = & \frac{G_N^2 m_1^2 m_2}{r^2} \left\{ v_1 \cdot a_1 \left[v_2^2 \left(\frac{28}{15} v_1^r + \frac{133}{10} v_2^r \right) - v_1 \cdot v_2 \left(\frac{56}{15} v_1^r + \frac{314}{15} v_2^r \right) \right] r \right. \\
& + v_2 \cdot a_1 \left[v_1^2 \left(\frac{47}{15} v_1^r - \frac{157}{15} v_2^r \right) + v_1 \cdot v_2 \left(\frac{229}{15} v_2^r - \frac{29}{15} v_1^r \right) \right. \\
& + v_2^2 \left(\frac{41}{20} v_1^r - \frac{133}{10} v_2^r \right) - \frac{103}{15} v_1^{r3} + \frac{304}{15} v_1^{r2} v_2^r - \frac{93}{5} v_1^r v_2^{r2} + \frac{133}{15} v_2^{r3} \Big] r \\
& + a_1^r \left[\frac{49}{6} v_1^4 + \frac{117}{10} (v_1 \cdot v_2)^2 + \frac{469}{80} v_2^4 + \frac{293}{30} v_1^2 v_2^2 - \frac{293}{15} v_1^2 v_1 \cdot v_2 - \frac{937}{60} v_1 \cdot v_2 v_2^2 \right. \\
& + v_1^2 \left(\frac{46}{3} v_1^r v_2^r - \frac{23}{3} v_1^{r2} - \frac{38}{3} v_2^{r2} \right) + v_1 \cdot v_2 \left(\frac{148}{15} v_1^r v_2^r - \frac{79}{15} v_1^{r2} + \frac{101}{15} v_2^{r2} \right) \\
& + v_2^2 \left(\frac{79}{30} v_1^{r2} - \frac{8}{5} v_1^r v_2^r - \frac{67}{10} v_2^{r2} \right) + v_2^r \left(\frac{332}{15} v_1^{r3} - \frac{383}{15} v_1^{r2} v_2^r + \frac{112}{15} v_1^r v_2^{r2} + \frac{22}{15} v_2^{r3} \Big] r \\
& + a_2^r v_1^2 \left[\frac{34}{5} v_1^{r2} - \frac{229}{15} v_1^r v_2^r + \frac{133}{10} v_2^{r2} - \frac{121}{30} v_1^4 \right] r + \frac{181}{30} v_1^6 - 8 (v_1 \cdot v_2)^3 + \frac{115}{32} v_2^6 + \frac{707}{60} v_1^4 v_2^2 \\
& - \frac{707}{30} v_1^4 v_1 \cdot v_2 + \frac{43}{2} (v_1 \cdot v_2)^2 v_1^2 + \frac{67}{4} (v_1 \cdot v_2)^2 v_2^2 + \frac{105}{16} v_2^4 v_1^2 - \frac{105}{8} v_1 \cdot v_2 v_2^4 - \frac{43}{2} v_1^2 v_1 \cdot v_2 v_2^2 \\
& + v_1^4 \left(\frac{326}{15} v_1^r v_2^r - \frac{163}{15} v_1^{r2} - \frac{157}{10} v_2^{r2} \right) + v_2^4 \left(\frac{3}{16} v_2^{r2} - \frac{21}{8} v_1^{r2} - \frac{3}{8} v_1^r v_2^r \right) - 9 (v_1 \cdot v_2)^2 v_1^{r2} \\
& + v_1^2 v_1 \cdot v_2 \left(\frac{128}{5} v_1^{r2} - \frac{418}{15} v_1^r v_2^r + \frac{314}{15} v_2^{r2} \right) + v_1^2 v_2^2 \left(\frac{443}{30} v_1^r v_2^r - \frac{64}{5} v_1^{r2} - \frac{133}{10} v_2^{r2} \right) \\
& + v_1 \cdot v_2 v_2^2 v_1^r \left(9 v_1^r + \frac{1}{2} v_2^r \right) + v_1^2 \left[\frac{68}{5} v_1^{r4} - \frac{484}{15} v_1^{r3} v_2^r + \frac{851}{15} v_1^{r2} v_2^{r2} - \frac{724}{15} v_1^r v_2^{r3} + \frac{266}{15} v_2^{r4} \right] \\
& - v_1 \cdot v_2 v_1^{r3} \left(\frac{241}{30} v_1^r + 2 v_2^r \right) + v_2^2 v_1^{r2} \left(\frac{5}{4} v_1^{r2} + v_1^r v_2^r - \frac{1}{2} v_2^{r2} \right) \\
& + v_1^{r4} \left(\frac{136}{25} v_1^r v_2^r + \frac{3}{2} v_2^{r2} - \frac{161}{25} v_1^{r2} \right) \Big\} + \mathcal{L}_{G_N^2}^{a^2}, \tag{17}
\end{aligned}$$

where terms involving more than one power of accelerations and their derivatives are:

$$\begin{aligned}
\mathcal{L}_{G_N^2}^{a^2} = & \frac{G_N^2 m_1^2 m_2}{r^2} \left\{ \left[\left(\frac{5317}{3600} - \frac{38}{15} \log \bar{r} \right) \dot{a}_1 \dot{a}_2 + \left(\frac{4}{15} \log \bar{r} - \frac{7}{75} \right) \dot{a}_1^r \dot{a}_2^r - \frac{1}{10} \dot{a}_1 \dot{a}_1 - \frac{14}{45} \dot{a}_1^r \dot{a}_1^r \right] r^2 \right. \\
& + \left[\left(\frac{68}{15} \log \bar{r} - \frac{332}{225} \right) v_{.a_1} \dot{a}_2^r - \frac{17}{3} v_{.a_1} \dot{a}_1^r + \left(\frac{24}{5} \log \bar{r} - \frac{923}{600} \right) v_{.a_2} \dot{a}_1^r \right] r \\
& + a_{2.} \dot{a}_1 \left[\left(\frac{24}{5} \log \bar{r} - \frac{283}{100} \right) v_1^r + \left(\frac{148}{15} \log \bar{r} - \frac{3581}{1800} \right) v_2^r \right] r - r a_2^r \dot{a}_1^r \left(\frac{6}{5} v_1^r + \frac{19}{20} v_2^r \right) \\
& + a_1^2 \left[\left(\frac{38}{15} \log \bar{r} - \frac{857}{225} \right) r a_2^r - \frac{39}{10} r a_1^r + \left(\frac{44}{3} \log \bar{r} - \frac{191}{18} \right) v_1^2 + \left(\frac{19}{18} - \frac{22}{3} \log \bar{r} \right) v_1 \cdot v_2 \right. \\
& + \left. \left(\frac{11}{3} \log \bar{r} - \frac{19}{36} \right) v_2^2 - \frac{15}{2} v_1^{r2} + \frac{15}{2} v_1^r v_2^r - \frac{23}{6} v_2^{r2} \right] + a_{1.} a_2 \left[\left(\frac{28}{3} \log \bar{r} + \frac{377}{72} \right) r a_1^r + \right. \\
& + \left. \left(\frac{332}{225} - \frac{68}{15} \log \bar{r} \right) r a_2^r + \left(\frac{268}{15} \log \bar{r} - \frac{4211}{1800} \right) v_1^2 - \left(\frac{32}{5} \log \bar{r} + \frac{124}{25} \right) v_1 \cdot v_2 \right. \\
& + \left. \left(\frac{158}{15} \log \bar{r} + \frac{253}{3600} \right) v_2^2 + \frac{241}{30} v_1^{r2} - \frac{107}{30} v_1^r v_2^r + \frac{277}{60} v_2^{r2} \right] - \frac{2}{3} r a_1^{r2} a_2^r + a_1^{r2} \left[\frac{1}{6} v_1^2 + \frac{2}{3} v_1 \cdot v_2 \right. \\
& - \left. \frac{1}{3} v_2^2 + \frac{3}{2} v_1^{r2} - 3 v_1^r v_2^r + \frac{7}{6} v_2^{r2} \right] + a_1^r a_2^r \left[\frac{59}{15} v_1 \cdot v_2 - \frac{19}{5} v_1^2 - \frac{173}{60} v_2^2 + \frac{11}{10} v_2^{r2} + \frac{67}{15} v_1^r v_2^r - \frac{56}{15} v_1^{r2} \right] \\
& + a_1^r \left[\frac{2}{3} v_{2.} a_1 v_1^r - \frac{2}{3} v_{.a_1} v_1^r + v_{.a_2} \left(\frac{31}{15} v_1^r + \frac{59}{15} v_2^r \right) - \frac{11}{6} v_{2.} a_2 v_2^r \right] \\
& + a_2^r \left[v_{.a_1} \left(\frac{94}{15} v_1^r - \frac{413}{30} v_2^r \right) - \frac{11}{6} v_{2.} a_1 v_1^r \right] + \left(\frac{22}{3} \log \bar{r} - \frac{158}{9} \right) (v_1 \cdot a_1)^2 + \frac{65}{3} v_1 \cdot a_1 v_{2.} a_1 \\
& - \frac{65}{6} (v_{2.} a_1)^2 + \left(\frac{128}{5} \log \bar{r} - \frac{267}{200} \right) v_{.a_1} v_{.a_2} + \left(\frac{44}{3} \log \bar{r} - \frac{347}{72} \right) v_{2.} a_1 v_{1.} a_2 \left. \right\} + \frac{\mathcal{L}^{pole}}{(d-3)}. \quad (18)
\end{aligned}$$

We write separately the divergent terms:

$$\begin{aligned}
\mathcal{L}^{pole} = & - (G_N^2 m_1^2 m_2) \left\{ \left[\frac{2}{15} \dot{a}_1^r \dot{a}_2^r - \frac{19}{15} \dot{a}_1 \cdot \dot{a}_2 \right] r^2 + \left[\left(\frac{34}{15} v_{.a_1} \dot{a}_2^r + \frac{12}{5} v_{.a_2} \dot{a}_1^r \right) + a_{1.} \dot{a}_1 \left(\frac{71}{30} v_1^r - \frac{11}{15} v_2^r \right) \right. \right. \\
& + a_{2.} \dot{a}_1 \left(\frac{12}{5} v_1^r + \frac{74}{15} v_2^r \right) \left. \right] r + \left[\frac{71}{60} r a_1^r + \frac{9}{10} r a_2^r + \frac{511}{60} v_1^2 - \frac{313}{60} v_1 \cdot v_2 + \frac{11}{5} v_2^2 \right] a_1^2 + \frac{11}{3} (v_1 \cdot a_1)^2 \\
& + \left[\frac{14}{3} r a_1^r - \frac{34}{15} r a_2^r + \frac{134}{15} v_1^2 - \frac{16}{5} v_1 \cdot v_2 + \frac{79}{15} v_2^2 \right] a_{1.} a_2 + \frac{64}{5} v_{.a_1} v_{.a_2} + \frac{22}{3} v_{1.} a_2 v_{2.} a_1 \left. \right\}, \quad (19)
\end{aligned}$$

with $v \equiv v_1 - v_2$ and $\bar{r} \equiv \sqrt{4\pi e^\gamma} \frac{r}{L}$ ($\gamma \simeq .577216$ is the Euler-Mascheroni constant). $\mathcal{L}_{G_N^2}^{a^2}$ is not relevant to determine the 4PN Lagrangian up to G_N^2 because it gives rise to $\mathcal{O}(G_N^3)$ terms via the double zero trick. As to its divergent part \mathcal{L}^{pole} , it will have to be canceled by an appropriate worldline redefinition, analogously to what happens at 3PN [8], thus leaving a finite result in the $d \rightarrow 3$ limit. It is however not possible to determine the exact form of such redefinition at this point, as it must be further constrained by the knowledge of the poles occouring at higher powers of G_N .

IV. THE 4PN LAGRANGIAN LINEAR IN ACCELERATIONS UP TO G_N^2

Before we can write the Lagrangian linear in accelerations up to term of order G_N^2 , that is the 4PN extension of the standard formula reported in [1], we have to take into account of the manipulations that led to such standard formula. Indeed, to make it linear in the accelerations, the double zero trick have repeatedly used, and we must compute the effects of such transformation at 4PN.

A. 2PN double zeros evaluated on 2PN equations of motion

The following 2PN term quadratic in the accelerations

$$\mathcal{L}^{a^2-2PN} = \frac{G_N m_1 m_2 r}{16} (15 a_1 \cdot a_2 - a_1^r a_2^r) \quad (20)$$

produces the 2PN and 3PN contributions correctly included in the standard 3PN Lagrangian, and the following 4PN term:

$$\begin{aligned} \left(\mathcal{L}_{4PN}^{a^2-2PN} \right)_{z^2} = & \frac{G_N^2 m_1^2 m_2}{r^2} \left\{ v_2 \cdot a_1 \left[v_1^2 \left(\frac{75}{8} v_1^r - \frac{15}{2} v_2^r \right) - \frac{15}{2} v_1 \cdot v_2 v^r - \frac{15}{2} v_2^2 v_1^r + \left(\frac{45}{4} v_2^r - \frac{135}{16} v_1^r \right) v_1^{r^2} \right] r \right. \\ & + v_1 \cdot a_1 \left(\frac{15}{2} v_1 \cdot v_2 v^r + \frac{45}{8} v_2^2 v_1^r \right) r + a_1^r \left[\frac{87}{32} v_1^4 + \frac{7}{2} (v_1 \cdot v_2)^2 - 7 v_1^2 v_1 \cdot v_2 + v_2^2 \left(\frac{1}{8} v_2^r - \frac{11}{4} v_1^r \right) v_1^r \right. \\ & + v_1^2 \left(\frac{1}{2} v_2^{r^2} - \frac{81}{8} v_1^r v_2^r - \frac{637}{32} v_1^{r^2} \right) - v_1 \cdot v_2 \cdot \left(10 v_1^{r^2} + v_1^r v_2^r - \frac{1}{2} v_2^{r^2} \right) + \left(\frac{21}{16} v_1^r - \frac{3}{4} v_2^r \right) v_1^{r^2} v_2^r \left. \right] \\ & + a_2^r v_1^2 \left(\frac{45}{8} v_1^{r^2} - \frac{15}{8} v_1^2 \right) + \frac{75}{32} v_1^6 - \frac{135}{32} v_1^4 v_1 \cdot v_2 + \frac{15}{8} v_1^4 v_2^2 + v_1^4 \left(\frac{315}{16} v_1^r v_2^r - \frac{555}{32} v_1^{r^2} - \frac{15}{4} v_2^{r^2} \right) \\ & + v_1^2 v_1 \cdot v_2 \left(\frac{585}{32} v_1^r - \frac{45}{4} v_2^r \right) v_1^r + \frac{45}{8} v_1^2 v_2^2 v_1^{r^2} + v_1^2 \left(\frac{453}{32} v_1^{r^2} - \frac{315}{8} v_1^r v_2^r + \frac{45}{2} v_2^{r^2} \right) v_1^{r^2} \\ & \left. + \frac{87}{32} v_1 \cdot v_2 v_1^{r^4} + \frac{261}{80} v^r v_1^{r^5} \right\} + \mathcal{O}(G_N^3). \quad (21) \end{aligned}$$

B. 3PN double zeros evaluated on 1PN equations of motion

The following 3PN term quadratic in accelerations

$$\begin{aligned}
\mathcal{L}^{a^2-3PN} = & (G_N m_1 m_2) r \left\{ \left(\frac{1}{96} \dot{a}_1^r \dot{a}_2^r - \frac{23}{288} \dot{a}_1 \cdot \dot{a}_2 \right) r^2 + \left(\frac{15}{32} a_2 \cdot \dot{a}_1 - \frac{1}{32} a_2^r \dot{a}_1^r \right) (v_1^r + v_2^r) r + \frac{7}{16} r v \cdot a_2 \dot{a}_1^r \right. \\
& - \frac{r}{32} a_1^{r2} a_2^r + a_1 \cdot a_2 \left(\frac{29}{32} r a_1^r - \frac{5}{16} v_1 \cdot v_2 + \frac{55}{32} v_1^2 + \frac{29}{32} v_1^{r2} + \frac{7}{16} v_1^r v_2^r \right) \\
& + a_1^r a_2^r \left(\frac{3}{16} v_1 \cdot v_2 - \frac{9}{32} v_1^2 + \frac{1}{16} v_1^r v_2^r + \frac{3}{32} v_1^{r2} \right) + a_1^r v \cdot a_2 \left(\frac{41}{32} v_1^r + \frac{13}{32} v_2^r \right) - \frac{1}{8} a_1^r v_2 \cdot a_2 v_2^r \\
& \left. + \frac{47}{16} v_1 \cdot a_1 v_1 \cdot a_2 - \frac{47}{32} v_1 \cdot a_1 v_2 \cdot a_2 - \frac{17}{32} v_2 \cdot a_1 v_1 \cdot a_2 \right\} \quad (22)
\end{aligned}$$

generate (as well as several 3PN terms) the following contribution:

$$\begin{aligned}
\left(\mathcal{L}_{4PN}^{a^2-3PN} \right)_{z^2} = & \frac{G_N^2 m_1^2 m_2}{r^2} \left\{ v_1 \cdot a_1 \left[v_1 \cdot v_2 \left(\frac{245}{24} v_1^r - \frac{271}{24} v_2^r \right) + v_2^2 \left(\frac{407}{24} v_2^r - \frac{71}{8} v_1^r \right) \right] r \right. \\
& + v_2 \cdot a_1 \left[v_1^2 \left(\frac{355}{48} v_1^r - \frac{35}{24} v_2^r \right) + v_1 \cdot v_2 \left(\frac{91}{24} v_2^r - \frac{139}{12} v_1^r \right) + v_2^2 \left(\frac{85}{8} v_1^r - \frac{407}{24} v_1^r \right) + \frac{61}{24} v_1^{r3} \right. \\
& - \frac{183}{16} v_1^{r2} v_2^r - \frac{181}{48} v_1^r v_2^{r2} + \frac{23}{12} v_2^{r3} \left. \right] r + a_1^r \left[\frac{689}{192} v_1^4 + \frac{17}{6} v_2^4 - \frac{21}{8} v_1^2 v_1 \cdot v_2 + \frac{16}{3} v_1^2 v_2^2 - \frac{75}{8} v_1 \cdot v_2 v_2^2 \right. \\
& + v_1^2 \left(\frac{967}{48} v_1^{r2} - \frac{55}{6} v_1^r v_2^r - \frac{337}{96} v_2^{r2} \right) + v_1 \cdot v_2 \left(\frac{67}{24} v_2^{r2} - \frac{31}{6} v_1^r v_2^r - \frac{253}{12} v_1^{r2} \right) \\
& + v_2^2 \left(\frac{35}{6} v_1^r v_2^r - \frac{3}{2} v_1^{r2} - \frac{2}{3} v_2^{r2} \right) + \left(\frac{439}{48} v_1^{r3} + \frac{193}{48} v_1^{r2} v_2^r - \frac{33}{16} v_1^r v_2^{r2} - \frac{1}{4} v_2^{r3} \right) v_2^r \left. \right] \\
& + a_2^r v_1^2 \left(\frac{23}{8} v_2^{r2} - \frac{205}{48} v_1^r v_2^r - \frac{169}{32} v_1^{r2} - \frac{215}{96} v_1^2 \right) + \frac{457}{192} v_1^6 - \frac{887}{192} v_1^4 v_1 \cdot v_2 + \frac{215}{96} v_1^4 v_2^2 \\
& + v_1^4 \left(\frac{233}{96} v_1^{r2} - \frac{127}{96} v_1^r v_2^r - \frac{635}{96} v_2^{r2} \right) + v_1^2 v_1 \cdot v_2 \left(\frac{481}{96} v_2^{r2} + \frac{151}{24} v_1^r v_2^r - \frac{339}{32} v_1^{r2} \right) \\
& + v_1^2 v_2^2 \left(\frac{169}{32} v_1^{r2} + \frac{205}{48} v_1^r v_2^r - \frac{23}{8} v_2^{r2} \right) + \frac{91}{6} v_1 \cdot v_2 v_1^{r4} \\
& + v_1^2 \left(\frac{67}{12} v_1^{r4} + \frac{737}{24} v_1^{r3} v_2^r - \frac{151}{12} v_1^{r2} v_2^{r2} - \frac{99}{8} v_1^r v_2^{r2} + \frac{23}{4} v_2^{r4} \right) - \frac{91}{5} v^r v_1^{r5} \left. \right\} + \mathcal{O}(G_N^3). \quad (23)
\end{aligned}$$

C. 4PN double zeroes coming from G1 graphs

Finally, acting with the double zero trick on the $\mathcal{L}_{G_N}^{a^2}$ reported in eq.(14) gives

$$\begin{aligned}
\left(\mathcal{L}_{G_N}^{a^2}\right)_{z^2} = & \frac{G_N^2 m_1^2 m_2}{r^2} \left\{ v_1 \cdot a_1 \left[v_1 \cdot v_2 \left(\frac{65}{48} v_1^r + \frac{137}{48} v_2^r \right) - v_2^2 \left(\frac{81}{32} v_1^r + \frac{665}{96} v_2^r \right) \right] r \right. \\
& + v_2 \cdot a_1 \left[v_1^2 \left(\frac{95}{96} v_2^r - \frac{65}{96} v_1^r \right) + v_1 \cdot v_2 \left(\frac{103}{48} v_1^r + \frac{25}{48} v_2^r \right) + v_2^2 \left(\frac{113}{32} v_1^r + \frac{665}{96} v_2^r \right) \right. \\
& + \left. \frac{557}{96} v_1^3 + \frac{39}{32} v_1^{r2} v_2^r - \frac{121}{96} v_1^r v_2^{r2} - \frac{37}{32} v_2^{r3} \right] r \\
& + a_1^r \left[\frac{185}{384} v_1^4 + \frac{23}{16} (v_1 \cdot v_2)^2 + \frac{89}{192} v_2^4 - \frac{13}{16} v_1^2 v_1 \cdot v_2 - \frac{11}{96} v_1^2 v_2^2 + \frac{27}{16} v_1 \cdot v_2 v_2^2 \right. \\
& + v_1^2 \left(\frac{499}{96} v_1^r v_2^r - \frac{673}{192} v_1^{r2} - \frac{49}{192} v_2^{r2} \right) + v_1 \cdot v_2 \left(\frac{52}{3} v_1^{r2} - \frac{73}{12} v_1^r v_2^r - \frac{19}{24} v_2^{r2} \right) \\
& + v_2^2 \left(\frac{9}{8} v_1^{r2} - \frac{71}{24} v_1^r v_2^r + \frac{5}{12} v_2^{r2} \right) + v_2^r \left(\frac{805}{96} v_1^{r2} v_2^r + \frac{1}{6} v_1^r v_2^{r2} - \frac{375}{16} v_1^{r3} - \frac{7}{192} v_2^{r3} \right) \left. \right] r \\
& + a_2^r v_1^2 \left[\frac{419}{384} v_1^2 + \frac{147}{64} v_1^{r2} - \frac{121}{96} v_1^r v_2^r - \frac{111}{64} v_2^{r2} \right] r - \frac{53}{384} v_1^6 + \frac{59}{48} v_1^4 v_1 \cdot v_2 - \frac{419}{384} v_1^4 v_2^2 \\
& + v_1^4 \left(\frac{149}{96} v_2^{r2} + \frac{205}{96} v_1^r v_2^r - \frac{19}{48} v_1^{r2} \right) + v_1^2 v_1 \cdot v_2 \left(\frac{95}{32} v_1^{r2} - \frac{281}{48} v_1^r v_2^r - \frac{53}{48} v_2^{r2} \right) \\
& + v_1^2 v_2^2 \left(\frac{111}{64} v_2^{r2} + \frac{121}{96} v_1^r v_2^r - \frac{147}{64} v_1^{r2} \right) + \frac{2615}{192} v_1 \cdot v_2 v_1^{r4} \\
& + v_1^2 \left(\frac{281}{24} v_1^{r2} v_2^{r2} - \frac{2443}{192} v_1^{r4} - \frac{121}{12} v_1^{r3} v_2^r - \frac{5}{24} v_1^r v_2^{r3} - \frac{37}{16} v_2^{r4} \right) + \frac{523}{32} v_1^r v_1^{r5} \left. \right\} + \mathcal{O}(G_N^3). \tag{24}
\end{aligned}$$

D. Summing up

Summing up all the contributions, we have:

$$\mathcal{L}^{4PN} = \frac{7}{256} v_1^{10} + \hat{\mathcal{L}}_{G_N}^{4PN} + \hat{\mathcal{L}}_{G_N^2}^{4PN} + \mathcal{O}(G_N^3), \tag{25}$$

where $\hat{\mathcal{L}}_{G_N}^{4PN} \equiv \mathcal{L}_{G_N}^{4PN} - \mathcal{L}_{G_N}^{a^2}$ can be read directly in eq.(13), while $\hat{\mathcal{L}}_{G_N}^{4PN}$ takes into account all the contributions previously discussed, and reads:

$$\begin{aligned}
\hat{\mathcal{L}}_{G_N}^{4PN} = & \frac{G_N^2 m_1^2 m_2}{r^2} \left\{ v_1 \cdot a_1 \left[v_2^2 \left(\frac{3733}{160} v_2^r - \frac{3769}{480} v_2^r \right) + v_1 \cdot v_2 \left(\frac{3769}{240} v_1^r - \frac{8849}{480} v_2^r \right) \right] r \right. \\
& + v_2 \cdot a_1 \left[v_1^2 \left(\frac{9229}{480} v_1^r - \frac{8849}{480} v_2^r \right) + v_1 \cdot v_2 \left(\frac{6499}{240} v_2^r - \frac{4529}{240} v_1^r \right) \right. \\
& + v_2^2 \left(\frac{2293}{160} v_1^r - \frac{3733}{160} v_2^r \right) - \frac{3341}{480} v_1^{r3} + \frac{10223}{480} v_1^{r2} v_2^r - \frac{3781}{160} v_1^r v_2^{r2} + \frac{4621}{480} v_2^{r3} \Big] r \\
& + a_1^r \left[\frac{6943}{384} v_1^4 + \frac{1331}{80} (v_1 \cdot v_2)^2 + \frac{2931}{320} v_2^4 + \frac{7193}{480} v_1^2 v_2^2 - \frac{7193}{240} v_1^2 v_1 \cdot v_2 - \frac{5593}{240} v_1 \cdot v_2 v_2^2 \right. \\
& + v_1^2 \left(\frac{2063}{96} v_1^r v_2^r - \frac{2099}{192} v_1^{r2} - \frac{3059}{192} v_2^{r2} \right) + v_1 \cdot v_2 \left(\frac{59}{60} v_1^{r2} - \frac{23}{60} v_1^r v_2^r + \frac{247}{30} v_2^{r2} \right) \\
& + v_2^2 \left(\frac{7}{5} v_1^r v_2^r - \frac{59}{120} v_1^{r2} - \frac{139}{20} v_2^{r2} \right) + v_2^r \left(\frac{2197}{240} v_1^{r3} - \frac{6661}{480} v_1^{r2} v_2^r + \frac{1337}{240} v_1^r v_2^{r2} + \frac{1133}{960} v_2^{r3} \Big] r \\
& + a_2^r v_1^2 \left[\frac{3021}{320} v_1^{r2} - \frac{9983}{480} v_1^r v_2^r + \frac{4621}{320} v_2^{r2} - \frac{13549}{1920} v_1^4 \right] r + \frac{20389}{1920} v_1^6 - 8 (v_1 \cdot v_2)^3 + \frac{115}{32} v_2^6 \\
& + \frac{28429}{1920} v_1^4 v_2^2 - \frac{29929}{960} v_1^4 v_1 \cdot v_2 + \frac{43}{2} (v_1 \cdot v_2)^2 v_1^2 + \frac{67}{4} (v_1 \cdot v_2)^2 v_2^2 + \frac{105}{16} v_2^4 v_1^2 - \frac{105}{8} v_1 \cdot v_2 v_2^4 \\
& - \frac{43}{2} v_1^2 v_1 \cdot v_2 v_2^2 + v_1^4 \left(\frac{1267}{30} v_1^r v_2^r - \frac{6283}{240} v_1^{r2} - \frac{1961}{80} v_2^{r2} \right) + v_2^4 \left(\frac{3}{16} v_2^{r2} - \frac{21}{8} v_1^{r2} - \frac{3}{8} v_1^r v_2^r \right) \\
& - 9 (v_1 \cdot v_2)^2 v_1^{r2} + v_1^2 v_1 \cdot v_2 \left(\frac{5501}{160} v_1^{r2} - \frac{9283}{240} v_1^r v_2^r + \frac{11923}{480} v_2^{r2} \right) + v_1 \cdot v_2 v_2^2 v_1^r \left(9 v_1^r + \frac{1}{2} v_2^r \right) \\
& + v_1^2 v_2^2 \left(\frac{9743}{480} v_1^r v_2^r - \frac{4941}{320} v_1^{r2} - \frac{4621}{320} v_2^{r2} \right) + v_2^2 v_1^{r2} \left(\frac{5}{4} v_1^{r2} + v_1^r v_2^r - \frac{1}{2} v_2^{r2} \right) \\
& + v_1^2 \left[\frac{6597}{320} v_1^{r4} - \frac{3061}{60} v_1^{r3} v_2^r + \frac{9403}{120} v_1^{r2} v_2^{r2} - \frac{1217}{20} v_1^r v_2^{r3} + \frac{4621}{240} v_2^{r4} \right] - v_1 \cdot v_2 v_1^{r3} \left(\frac{6587}{960} v_1^r + 2 v_2^r \right) \\
& + v_1^{r4} \left(\frac{3227}{800} v_1^r v_2^r + \frac{3}{2} v_2^{r2} - \frac{4027}{800} v_1^{r2} \right) \Big\} . \tag{26}
\end{aligned}$$

The above formula, along with the term explicitly written in equation (13), is the main result of this paper.

V. ENERGY

The following expression for the Energy is derived from the Lagrangian by means of a generalized Legendre transform :

$$\begin{aligned}
E^{4PN} = & \frac{63}{256}v_1^{10} + \frac{G_N m_1 m_2}{r} \left\{ \frac{525}{128}v_1^8 + \frac{1}{32}(v_1.v_2)^4 - \frac{1291}{128}v_1^6 v_1.v_2 + \frac{731}{128}v_1^6 v_2^2 + \frac{3}{21}v_1^2 (v_1.v_2)^3 \right. \\
& + \frac{373}{64}v_1^4 (v_1.v_2)^2 + \frac{765}{256}v_1^4 v_2^4 - \frac{1511}{128}v_1^4 v_1.v_2 v_2^2 + 3v_1^2 (v_1.v_2)^2 v_2^2 + \frac{3}{32}(v_1.v_2)^3 v_1^{r2} \\
& - v_1^6 \left(\frac{181}{128}v_1^r + \frac{295}{128}v_2^r \right) v_2^r + v_1^4 v_1.v_2 \left(\frac{35}{16}v_1^{r2} + \frac{35}{32}v_1^r v_2^r + \frac{21}{4}v_2^{r2} \right) \\
& - v_1^4 v_2^2 \left(\frac{501}{128}v_1^{r2} + \frac{185}{128}v_1^r v_2^r + \frac{231}{64}v_2^{r2} \right) + v_1^2 (v_1.v_2)^2 \left(\frac{51}{32}v_1^{r2} + \frac{23}{32}v_1^r v_2^r - \frac{21}{16}v_2^{r2} \right) \\
& + v_1^2 v_1.v_2 v_2^2 \left(\frac{333}{64}v_1^r + \frac{3}{8}v_2^r \right) v_1^r + v_1^4 \left(\frac{267}{128}v_1^{r3} + \frac{159}{128}v_1^{r2} v_2^r + \frac{93}{128}v_1^r v_2^{r2} + \frac{183}{128}v_2^{r3} \right) v_2^r \\
& + (v_1.v_2)^2 \left(\frac{31}{64}v_1^{r2} - \frac{25}{32}v_1^r v_2^r - \frac{15}{32}v_2^{r2} \right) v_1^{r2} + v_1^2 v_2^2 \left(\frac{411}{128}v_1^{r2} + \frac{99}{64}v_1^r v_2^r + \frac{3}{4}v_2^{r2} \right) v_1^{r2} \\
& - v_1^2 v_1.v_2 \left(\frac{411}{128}v_1^{r4} - \frac{25}{16}v_1^{r3} v_2^r - \frac{45}{64}v_1^{r2} v_2^{r2} - \frac{3}{8}v_1^r v_2^{r3} - \frac{245}{128}v_2^{r4} \right) + v_1.v_2 \left(\frac{125}{128}v_1^{r3} + \frac{25}{32}v_1^{r2} v_2^r \right. \\
& + \left. \frac{85}{128}v_1^r v_2^{r2} + \frac{5}{16}v_2^{r3} \right) v_1^{r3} - v_1^2 \left(\frac{165}{128}v_1^{r5} + \frac{135}{128}v_1^{r4} v_2^r + \frac{55}{64}v_1^{r3} v_2^{r2} + \frac{45}{64}v_1^{r2} v_2^{r3} + \frac{75}{128}v_1^r v_2^{r4} \right. \\
& + \left. \frac{125}{128}v_2^{r5} \right) v_2^r + \left(\frac{35}{128}v_1^{r3} + \frac{25}{128}v_1^{r2} v_2^r + \frac{35}{128}v_1^r v_2^{r2} + \frac{35}{256}v_2^{r3} \right) v_1^{r4} v_2^r \Big\} \\
& + \frac{G_N^2 m_1^2 m_2}{r^2} \left\{ \frac{27}{2}v_1^6 - \frac{121}{2}(v_1.v_2)^3 - \frac{537}{32}v_2^6 - \frac{153}{2}v_1^4 v_1.v_2 + \frac{175}{4}v_1^4 v_2^2 + \frac{247}{2}v_1^2 (v_1.v_2)^2 \right. \\
& + \frac{219}{2}(v_1.v_2)^2 v_2^2 + \frac{661}{16}v_1^2 v_2^4 - \frac{593}{8}v_1.v_2 v_2^4 - \frac{553}{4}v_1^2 v_1.v_2 v_2^2 + v_1^4 \left(\frac{4779}{64}v_1^r v_2^r - \frac{1275}{64}v_1^{r2} - \frac{47}{4}v_2^{r2} \right) \\
& + (v_1.v_2)^2 \left(\frac{1977}{16}v_1^r v_2^r - \frac{1487}{16}v_1^{r2} - 55v_2^{r2} \right) + v_2^4 \left(\frac{39}{64}v_1^{r2} - \frac{855}{64}v_1^r v_2^r + \frac{15}{16}v_2^{r2} \right) \\
& + v_1^2 v_1.v_2 \left(\frac{39}{64}v_1^{r2} - \frac{855}{64}v_1^r v_2^r + \frac{267}{4}v_2^{r2} \right) + v_1^2 v_2^2 \left(\frac{2321}{32}v_1^r v_2^r - \frac{1111}{32}v_1^{r2} - \frac{151}{8}v_2^{r2} \right) \\
& + v_1.v_2 v_2^2 \left(\frac{463}{8}v_1^{r2} - \frac{979}{16}v_1^r v_2^r + \frac{151}{8}v_2^{r2} \right) + v_1^2 \left(\frac{3541}{32}v_2^{r3} v_1^r + \frac{4147}{32}v_1^{r2} v_2^{r2} - \frac{10507}{96}v_1^{r3} v_2^r \right. \\
& - \frac{125}{24}v_1^{r4} + \frac{295}{12}v_2^{r4} \Big) + v_1.v_2 \left(\frac{8249}{48}v_1^{r3} v_2^r + \frac{1493}{16}v_1^{r2} v_2^{r2} + \frac{5351}{48}v_1^r v_2^{r3} - \frac{295}{12}v_2^{r4} - \frac{3317}{96}v_1^{r4} \right) \\
& + v_2^2 \left(\frac{1001}{96}v_1^{r3} - \frac{1189}{32}v_1^{r2} v_2^r - \frac{619}{32}v_1^r v_2^{r2} + \frac{2155}{96}v_2^{r3} \right) v_1^r + \left(\frac{4657}{64}v_1^{r4} v_2^r - \frac{49}{320}v_1^{r5} \right. \\
& \left. - \frac{4269}{32}v_1^{r3} v_2^{r2} + \frac{3183}{32}v_1^{r2} v_2^{r3} - \frac{1715}{32}v_1^r v_2^{r4} - \frac{1621}{320}v_2^{r5} \right) \Big\} + \mathcal{O}(G_N^3). \tag{27}
\end{aligned}$$

In the above expression, which is conserved up to order G^2 if one neglects gravitational radiation, accelerations appearing at 4PN have been systematically substituted by their

Newtonian value, and we included the post-Newtonian corrections of the analogous substitutions done in [10] to write the 3PN Energy.

VI. LORENTZ INVARIANCE AND CENTER-OF-MASS POSITION

A. Boosts

As discussed in [10], the post-Newtonian dynamics must be reminiscent of the symmetries of the fundamental theory. In practice, it must reflect the invariance under boosts, which in this framework are implemented by the following transformation

$$\delta \vec{x}_{1,2} = -\vec{V}t + (V \cdot x_{1,2}) \vec{v}_{1,2} + \mathcal{O}(V^2), \quad (28)$$

where V^i is the boost velocity.

One must then require that the equations of motion are invariant under boosts, that is the Lagrangian variation must consist only of a time derivative plus possibly double zero terms:

$$\delta \mathcal{L}^{4PN} = V \cdot \dot{Z}^{4PN} + \text{double zeroes}. \quad (29)$$

As the boost transformation does not depend on G_N (nor on d), one can to some extent check Lorentz invariance order by order in Newton's constant; we find indeed that the above

equation is compatible with $\vec{Z}^{4PN} = \frac{5}{128}v_1^8\vec{x}_1 + \vec{Z}_{G_N}^{4PN} + \vec{Z}_{G_N^2}^{4PN} + \mathcal{O}(G_N^3)$, with

$$\begin{aligned} \vec{Z}_{G_N}^{4PN} = & \frac{Gm_1m_2}{r} \left\{ \left[\frac{11}{4}v_1^4v_2^r + (v_1.v_2)^2 \left(\frac{1}{4}v_2^r - \frac{3}{16}v_1^r \right) + v_2^4 \left(\frac{49}{64}v_1^r + \frac{123}{64}v_2^r \right) - 3v_1^2v_1.v_2 \right. \right. \\ & - \frac{123}{32}v_1^2v_2^2 + v_1.v_2 \left(\frac{19}{32}v_1^rv_2^{r2} - \frac{5}{32}v_1^{r2}v_2^r + \frac{5}{12}v_2^{r3} - \frac{1}{16}v_1^{r3} \right) + v_2^2 \left(\frac{23}{32}v_2^r - \frac{77}{96}v_1^r \right) v_1^{r2} \\ & + \left. \left(\frac{13}{64}v_1^{r4} + \frac{11}{64}v_1^{r3}v_2^r + \frac{5}{32}v_1^{r2}v_2^{r2} + \frac{1}{8}v_1^rv_2^{r3} + \frac{1}{16}v_2^{r4} \right) v_1^r \right] \vec{v}_1 \\ & + \left[v_2.a_1 \left(\frac{1}{4}v_1.v_2 - v_1^2 - \frac{15}{8}v_2^2 + \frac{1}{8}v_1^rv_2^r + \frac{1}{16}v_2^{r2} \right) v_2^r + v_1.a_1 \left(\frac{11}{4}v_1^2 - 2v_1.v_2 - \frac{15}{8}v_2^2 \right) v_2^r \right. \\ & + a_1^r \left(\frac{3}{8}v_1.v_2v_2^{r2} - v_2^2 \left\{ \frac{5}{8}v_1^{r2} + \frac{1}{2}v_1^rv_2^r + \frac{5}{16}v_2^{r2} \right\} + \frac{1}{8}v_1^rv_2^{r3} + \frac{1}{16}v_2^{r4} \right) + \frac{11}{16}v_1^6 + \frac{1}{16}(v_1.v_2)^2 \\ & - \frac{15}{16}v_1^4v_2^2 + \frac{13}{16}v_1^4v_1.v_2 + v_1^2v_2^4 - \frac{1}{32}v_1^2v_1.v_2v_2^2 - v_1^4 \left(\frac{7}{16}v_2^r + \frac{11}{16}v_1^r \right) v_2^r + v_1^2v_1.v_2 \left(\frac{11}{8}v_2^r + v_1^r \right) v_2^r \\ & - v_1^2v_2^2 \left(\frac{5}{8}v_1^r + \frac{23}{32}v_2^r \right) v_1^r + (v_1.v_2)^2 \left(\frac{11}{64}v_2^{r2} - \frac{11}{64}v_1^{r2} + \frac{5}{316}v_1^rv_2^r \right) - v_1.v_2v_2^2v_1^{r2} + \frac{11}{16}v_2^4v_1^{r2} \\ & + \left. v_1^2 \left(\frac{1}{8}v_1^r + \frac{5}{8}v_2^r \right) v_2^{r3} + v_1.v_2 \left(\frac{19}{16}v_1^{r4} + v_1^{r3}v_2^r - \frac{15}{32}v_1^{r2}v_2^{r2} - v_1^rv_2^{r3} - \frac{19}{16}v_2^{r4} \right) - \frac{5}{23}v_1^{r3}v_2^{r3} \right] \vec{x}_1 \Big\} \end{aligned} \quad (30)$$

and

$$\begin{aligned} \vec{Z}_{G_N^2}^{4PN} = & \frac{G^2m_1m_2}{r^2} \left\{ m_2 \left[\frac{235}{24}v_1.v_2v_1^r + \frac{235}{48}v_2^2v_1^r + \frac{29}{24}v_1^rv_2^{r2} \right] \vec{v}_1 \right. \\ & + m_1 \left[v_1^2 \left(\frac{3679}{480}v_1^r - \frac{8849}{480}v_2^r \right) + \frac{235}{12}v_1.v_2v_1^r - \frac{3341}{480}v_1^{r3} + \frac{10223}{480}v_1^{r2}v_2^r - \frac{9983}{480}v_1^rv_2^{r2} \frac{4621}{480}v_2^{r3} \right] \vec{v}_1 \\ & + m_1 \left[\frac{235}{24}v_2.a_1v_2^r + a_1^r \left(\frac{185}{8}v_1.v_2 - \frac{185}{16}v_1^2 - \frac{20}{3}v_2^2 + \frac{17}{6}v_2^{r2} \right) - \frac{149}{24}v_1^4 + \frac{235}{16}v_1^2v_2^2 \right. \\ & + v_1^2 \left(\frac{161}{24}v_1^{r2} - \frac{223}{24}v_1^rv_2^r + \frac{97}{24}v_2^{r2} \right) - \frac{147}{8}v_1.v_2v_1^{r2} + \left. \left(\frac{10841}{2880}v_1^r - \frac{29}{6}v_2^r \right) v_1^{r3} \right] \vec{x}_1 \\ & + m_2 \left[-\frac{235}{24}v_2.a_1v_2^r - a_1^r \left(\frac{235}{48}v_2^2 + \frac{29}{24}v_2^{r2} \right) - \frac{45}{16}v_1^4 - \frac{463}{24}v_1^2v_2^2 - \frac{25}{16}v_2^4 + \frac{19}{2}v_1^2v_1.v_2 - \frac{463}{48}v_1^2v_2^2 \right. \\ & + \frac{733}{48}v_1.v_2v_2^2 + v_1^2 \left(\frac{1}{2}v_1^rv_2^r - \frac{1}{4}v_1^{r2} + \frac{7}{24}v_2^{r2} \right) + v_2^2 \left(\frac{23}{4}v_1^{r2} - \frac{45}{8}v_1^rv_2^r + \frac{7}{2}v_2^{r2} \right) \\ & + \left. v_1.v_2 \left(\frac{97}{6}v_1^r - \frac{11}{3}v_2^r \right) v_2^r + \left(\frac{35}{6}v_1^{r2} + \frac{42}{9}v_1^rv_2^r - \frac{4187}{960}v_2^{r2} \right) v_2^{r2} \right] \vec{x}_1 \Big\} \end{aligned} \quad (31)$$

(we remind that sum under particle exchange $1 \leftrightarrow 2$ is understood).

More exactly, we find that eq.(29) holds up to order G_N if we assume that the following

terms

$$\begin{aligned}
& \frac{G_N m_1 m_2}{r} \left[a_1 \cdot a_2 \left(\frac{1}{4} v_1 \cdot v_2 - v_2^2 - \frac{15}{8} v_1^2 + \frac{3}{8} v_1^r v_2^r + \frac{5}{12} v_1^{r^2} \right) v_1^r + \frac{15}{4} v_1 \cdot a_1 v_2 \cdot a_2 v_1^r + \right. \\
& + \frac{1}{4} v_2 \cdot a_1 v_1 \cdot a_2 v_1^r + v_1 \cdot a_1 a_2^r \left(\frac{11}{4} v_1^2 - 2 v_1 \cdot v_2 - \frac{15}{8} v_2^2 + \frac{5}{8} v_1^{r^2} + v_1^r v_2^r + \frac{5}{4} v_2^r \right) \\
& + v_2 \cdot a_1 a_2^r \left(\frac{1}{4} v_1 \cdot v_2 - v_1^2 - \frac{15}{8} v_2^2 + \frac{3}{8} v_1^{r^2} + \frac{3}{4} v_1^r v_2^r + \frac{5}{4} v_2^r \right) \\
& \left. + a_1^r a_2^r \left(\frac{3}{4} v_1 \cdot v_2 - \frac{5}{8} v_1^2 - \frac{1}{2} v_2^2 + \frac{1}{4} v_1^{r^2} + \frac{3}{8} v_1^r v_2^r \right) v_1^r \right] \vec{r} \quad (32)
\end{aligned}$$

are part of the double zero structure indicated at the r.h.s. of the same equation. Although we cannot be sure that this is indeed the case without knowing also the G_N^3 part of the Lagrangian, it is a nontrivial fact that the only terms quadratic in the accelerations appear above; moreover, our assumption implies that such double zeroes must be accounted for when checking for the invariance at order G_N^2 and this is exactly what we found.

In its turn, Lorentz invariance at such order holds if we postulate the following $\mathcal{O}(G_N^2)$ double zero structure

$$\frac{G_N^2 m_1^2 m_2}{r^2} \left\{ \left[a_1^r a_2^r \left(\frac{29}{12} v_1^r - \frac{17}{3} v_2^r \right) + \frac{235}{24} \left(v_1 \cdot a_1 a_2^r + v_1^r a_1 \cdot a_2 - \frac{40}{3} v_1 \cdot a_2 a_1^r \right) \right] \vec{r} - \frac{235}{12} v_1 \cdot a_2 a_1^r \vec{x}_2 \right\}, \quad (33)$$

whose actual existence will be checked only after determination of the 4PN G_N^3 dynamics (again, it is encouraging that only terms quadratic in accelerations are left). It is worth noticing that in deriving these results we had to take into account the 1PN corrections to the double zero terms which are already present at 3PN.

B. Center of mass position

Still following [10], we can use the result of the previous subsection, and the fact of having a Lagrangian linear in acceleration, to determine the center-of-mass position at order G_N^2 .

$$\begin{aligned}
\vec{G}^{4PN} = & \frac{5}{128} m_1 v_1^8 \vec{x}_1 + \frac{G_N m_1 m_2}{r} \left\{ \left[\frac{11}{16} v_1^3 + \frac{1}{16} (v_1 \cdot v_2)^3 - \frac{13}{16} v_1^4 v_1 \cdot v_2 + v_1^4 v_2^2 - \frac{15}{16} v_1^2 v_2^4 - \frac{1}{32} v_1^2 v_1 \cdot v_2 v_2^2 \right. \right. \\
& + v_1^4 \left(\frac{11}{16} v_1^r - \frac{7}{16} v_2^r \right) v_1^r - \frac{11}{64} (v_1 \cdot v_2) (v_1^r + v_2^r) v^r + \frac{11}{16} v_2^4 v_1^{r2} + v_1^2 v_1 \cdot v_2 \left(v_1^r + \frac{11}{8} v_2^r \right) v_2^r \\
& - v_1^2 v_2^2 \left(\frac{5}{8} v_1^r + \frac{23}{32} v_2^r \right) v_1^r - v_1 \cdot v_2 v_2^2 v_1^{r2} + v_1 \cdot v_2 \left(\frac{19}{16} v_1^{r4} + v_1^{r3} v_2^r - \frac{15}{32} v_1^{r2} v_2^{r2} - v_1^r v_2^{r3} - \frac{19}{16} v_2^{r4} \right) \\
& + v_1^2 \left(\frac{1}{8} v_1^r + \frac{5}{8} v_2^r \right) v_2^{r3} - \frac{5}{32} v_1^{r3} v_2^{r3} \Big] \vec{x}_1 \\
& + \left[\frac{11}{4} v_1^4 v_2^r + (v_1 \cdot v_2)^2 \left(\frac{1}{4} v_2^r - \frac{3}{16} v_1^r \right) + v_2^4 \left(\frac{49}{64} v_1^r + \frac{123}{64} v_2^r \right) - 3 v_1^2 v_1 \cdot v_2 v_2^r - \frac{123}{32} v_1^2 v_2^2 v_1^r \right. \\
& - \frac{1}{16} v_1 \cdot v_2 v_2^2 v_1^r + v_2^2 \left(\frac{23}{32} v_2^r - \frac{77}{96} v_1^r \right) v_1^{r2} + v_1 \cdot v_2 \left(\frac{5}{12} v_2^{r3} + \frac{19}{32} v_2^{r2} v_1^r - \frac{5}{32} v_1^{r2} v_2^r - \frac{1}{16} v_1^{r3} \right) \\
& + \left. \left(\frac{13}{64} v_1^{r4} + \frac{11}{64} v_1^{r3} v_2^r + \frac{5}{32} v_1^{r2} v_2^{r2} + \frac{1}{8} v_1^r v_2^{r3} \frac{1}{16} v_2^{r4} \right) v_1^r \Big] \vec{v}_1 \right\} \\
& + \frac{G_N^2 m_1 m_2}{r^2} \left\{ m_2 \left[\frac{45}{16} v_1^4 + \frac{463}{23} (v_1 \cdot v_2)^2 - \frac{3}{16} v_2^4 - \frac{19}{2} v_1^2 v_1 \cdot v_2 + \frac{421}{8} v_1^2 v_2^2 - \frac{565}{48} v_1 \cdot v_2 v_2^2 \right. \right. \\
& + v_1^2 \left(\frac{1}{4} v_1^{r2} + \frac{13}{4} v_1^r v_2^r + \frac{31}{12} v_2^{r2} \right) + v_2^2 \left(\frac{9}{8} v_2^{r2} + \frac{41}{8} v_1^r v_2^r - \frac{13}{8} v_1^{r2} \right) + v_1 \cdot v_2 \left(\frac{19}{24} v_2^r - \frac{127}{6} v_1^r \right) v_2^r \\
& + v_1^2 \left(\frac{667}{960} v_2^{r2} + \frac{11}{72} v_1^r v_2^r - \frac{19}{3} v_1^{r2} \right) v_2^{r2} \Big] \vec{x}_1 + m_1 \left[\frac{149}{21} v_1^4 - \frac{235}{16} v_1^2 v_1 \cdot v_2 + \frac{147}{8} v_1 \cdot v_2 v_1^{r2} \right. \\
& + v_1^2 \left(\frac{223}{24} v_1^r v_2^r - \frac{161}{24} v_1^{r2} - \frac{97}{24} v_2^{r2} \right) + \left(\frac{29}{6} v_2^r - \frac{10481}{2880} v_1^r \right) v_1^{r3} \Big] \vec{x}_1 \\
& + m_1 \left[v_1^2 \left(\frac{8849}{480} v_2^r - \frac{3679}{480} v_1^r \right) - \frac{235}{12} v_1 \cdot v_2 v_1^r + \frac{3341}{480} v_1^{r3} - \frac{10223}{480} v_1^{r2} v_2^r + \frac{9983}{480} v_1^r v_2^{r2} - \frac{4621}{480} v_2^{r3} \right] \vec{v}_1 \\
& + m_2 \left[\frac{235}{24} v_1 \cdot v_2 v_2^r + \frac{235}{48} v_2^2 v_1^r + \frac{29}{24} v_1^r v_2^{r2} \right] \vec{v}_1 \Big\} + \mathcal{O}(G_N^3). \tag{34}
\end{aligned}$$

As for the Energy, accelerations appearing at 4PN have been systematically substituted by their Newtonian value, and we included the post-Newtonian corrections of the analogous substitutions done in [10] to derive 3PN the center-of-mass position.

VII. CONCLUSIONS

We have computed the the conservative dynamics of a binary system within the framework of the PN approximation to general relativity at fourth post-Newtonian order up to terms

quadratic in the Newton constant. By a systematic use of the effective field theory methods for non-relativistic general relativity proposed by Goldberger and Rothstein it has been possible to automatize the computation and compute the effective Lagrangean by summing the contributions of several Feynman diagrams. We have also computed the energy function and verified for Lorentz invariance, giving a non trivial consistency check of our calculations.

This work is the first ingredient for the complete determination of the 4PN dynamics.

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- [25] We adopt the “mostly plus” convention $\eta_{\mu\nu} \equiv \text{diag}(-, +, +, +)$, and the Riemann and Ricci tensors are defined as $R^\mu_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho} \Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma} \Gamma^\alpha_{\nu\rho}$, $R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu}$.

- [26] $\hat{\Gamma}_{jk}^i$ is the connection of the purely spatial metric γ_{ij} , $F_{ij} \equiv A_{j,i} - A_{i,j}$ and indices must be raised and contracted via the d -dimensional metric tensor γ , although in the last three lines of eq. 7 one can use just δ_{ij} as neglected terms are not needed at 4PN order; on the other hand all the spatial derivatives are meant to be simple (not covariant) ones and, when ambiguities might raise, gradients are always meant to act on contravariant fields (so that, for instance, $\vec{\nabla} \cdot \vec{A} \equiv \gamma^{ij} A_{i,j}$ and $F_{ij}^2 \equiv \gamma^{ik} \gamma^{jl} F_{ij} F_{kl}$).
- [27] As we focus on the conservative part of the dynamics, internal lines in Feynman diagrams correspond to the so-called *potential* gravitons and no gravitational radiation is emitted.